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THE FACE LATTICE OF HYPERPLANE ARRANGEMENTS

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Every arrangement $\mathscr{H}$ of affine hyperplanes in $\boldsymbol{R}^d$ determines a partition of $\boldsymbol{R}^d$ into open topological cells. The face lattice $L(\mathscr{H})$ of this partition was the object of a study by Barnabei and Brini, who determined the homotopy type of its intervals.

We use geometric constructions from the theory of convex polytopes to prove the shellability of $L(\mathscr{H})$ and to determine the combinatorial topology of its intervals up to homeomorphism.

1. Introduction

Let $\mathscr{H}$ be a finite set of affine hyperplanes in a euclidean vector space $\boldsymbol{R}^d$. This *arrangement* determines a partition $\Pi(\mathscr{H})$ of $\boldsymbol{R}^d$ into open topological cells, which can be described as follows: for each hyperplane $H \in \mathscr{H}$, let π_H be the partition of $\boldsymbol{R}^d$ into three parts given by H and the two open halfspaces determined by H. Then $\Pi(\mathscr{H})$ is the meet of the partitions π_H $(H \in \mathscr{H})$ in the lattice of all partitions of $\boldsymbol{R}^d$. The parts of $\Pi(\mathscr{H})$ are non-empty intersections of parts of the partitions π_H, hence open topological cells [12, p. 214] embedded convexly into $\boldsymbol{R}^d$.

Let $P(\mathscr{H})$ be the poset of these cells (i.e. the parts of $\Pi(\mathscr{H})$), ordered by inclusion of their closures. The minimal elements of $P(\mathscr{H})$ are all cells of the same dimension. By reduction to a suitable quotient space we can assume (without loss of generality for the study of $P(\mathscr{H})$) that the minimal elements of $P(\mathscr{H})$ are in fact points, i.e. that $\mathscr{H}$ is *regular* in the sense of [1, p. 114].

Now let $L(\mathscr{H})$ be the *face lattice* of the dissection of $\boldsymbol{R}^d$ by $\mathscr{H}$, that is, the poset $P(\mathscr{H})$ with a minimal element $\hat{0} = \emptyset$ and a maximal element $\hat{1} = \boldsymbol{R}^d$ adjoined. $L(\mathscr{H})$ is a finite graded lattice of length $d+2$.

The face lattice $L(\mathscr{H})$ of an arrangement was studied by Barnabei and Brini [1], who determined the homotopy type of its intervals.

However, adding the "point at infinity" to $\boldsymbol{R}^d$ (corresponding to one point compactification of $\boldsymbol{R}^d$), it is easy to see that $L(\mathscr{H})$ is the face lattice of a regular cell decomposition [12, p. 216] of S^d, from which an atom (corresponding to the point at infinity) has been deleted. Thus one might expect to find sharper results about the topology of $L(\mathscr{H})$.

In this note we will exhibit the underlying geometry of the problem and use it to determine the topology of the face lattice $L(\mathscr{H})$ up to homeomorphism.

To establish the shellability of $L(\mathcal{H})$, we use the famous construction of Bruggesser and Mani [5] showing the shellability of convex polytopes.

We assume the reader to be familiar with the notions of topology of posets as in [2], duality of polytopes and the geometry of zonotopes as in [8] or [10], and the Bruggesser–Mani argument in [5, 6] or [11].

2. Central arrangements

We start with a special case. In this section, $\mathcal{H}$ will always be a *central arrangement*, that is, $\mathcal{H} = \{H_1, \dots, H_n\}$ is a finite set of $(d-1)$-dimensional *linear* subspaces in $\boldsymbol{R}^d$. With the regularity condition discussed in Section 1, this means $\bigcap_{i=1}^n H_i = \{0\}$.

The following lemma is probably "folklore" in convex polytope theory, although we could not find it explicitly stated in the literature. Grünbaum [8, p. 409] even states it as an open problem. We will not only need the result as stated, but also special properties of the construction used for its proof. The crucial idea goes back to observations by McMullen [10, p. 94].

Lemma 2.1. *The cell decomposition of the unit sphere S^{d-1} induced by a central arrangement $\mathcal{H}$ is polytopal, that is, there exists a convex polytope whose boundary complex is combinatorially equivalent to it.*

Proof. Let $z_1, \dots, z_n$ be unit vectors in $\boldsymbol{R}^d$ orthogonal to the corresponding hyperplanes, such that $H_i = \{z_i\}^{\perp}$ for $1 \leq i \leq n$. Let $Z = Z(\mathcal{H})$ be the associated zonotope

$$Z = [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_n, z_n],$$

and consider

$$Z^* = \{x \in \boldsymbol{R}^d : \langle x, z\rangle \leq 1 \quad \text{for all} \quad z \in Z\},$$

its polar dual (as in [10, §2]).

The i-faces of Z^* are in one-to-one correspondence with the i-faces of the cell decomposition of S^{d-1} such that for every open i-face F of Z^* $(0 \leq i \leq d-1)$,

$$\boldsymbol{R}^+F = \{\lambda z : \lambda \in \boldsymbol{R}^+; z \in F\}$$

is an open $(i+1)$-cell in the decomposition of $\boldsymbol{R}^d$ by $\mathcal{H}$, and

$$S^{d-1} \cap \boldsymbol{R}^+F = \left\{\frac{z}{\|z\|} : z \in F\right\}$$

is the corresponding i-cell in S^{d-1}. This observation is implicit in [10, §2]. $\square$

Corollary 2.2. *For central hyperplane arrangements $\mathcal{H}$, the face lattice $L(\mathcal{H})$ is* *shellable. Its proper part is homeomorphic to a d-ball. In fact, it is the cone over a $(d-1)$-sphere.*

Proof. Let $L(Z^*)$ be the face lattice of Z^*. The proof of Lemma 2.1 shows $L(\mathscr{H}) = \{\hat{0}\} \,\dot{\cup}\, L(Z^*)$. Therefore $L(\mathscr{H})$ is shellable, because the shellability of convex polytopes [5] induces shellability of their face lattices [2, 3, 4]. Thus the order complex of

$$\overline{L(\mathscr{H})} = L(Z^*) \setminus \{\hat{1}\}$$

is a cone over the barycentric subdivision of the boundary complex of Z^* and hence homeomorphic to B^d. □

3. Affine arrangements

Now let $\mathscr{H}$ be an affine hyperplane arrangement in $\mathbb{R}^d$, not necessarily central. Both the shellability of $L(\mathscr{H})$ and the topology of the intervals in this case are most easily established by using a geometric construction that reduces the situation to the case of central arrangements solved in Section 2, and the following lemma.

Lemma 3.1. *Let K be a convex (bounded) d-polytope and $L(K)$ its face lattice. Then for every proper face $M \in \overline{L(K)}$ of M, the subposet*

$$\{F \in \overline{L(K)} : F \nleq M\}$$

of $L(K)$ is the dual of the face poset of a shellable $(d-1)$-ball. In particular, the poset is shellable and homeomorphic to a $(d-1)$-ball.

Proof. Taking duals, we see that

$$\{F \in \overline{L(K)} : F \nleq M\}^* = \{F \in \overline{L(K^*)} : F \ngeq \hat{M}\}$$

is the face poset of the boundary complex ∂K^* of K^*, with the star of $\hat{M}$ removed. (The star of $\hat{M}$ is the set of all faces containing $\hat{M}$ in their closures.) Now by an adaption [6, Theorem 3.2] of the Bruggesser–Mani argument we can choose a shelling of ∂K^* in which the facets in star$(\hat{M})$ come last. (Shellings of polyhedral complexes are defined in [5, 6] and [11]. We work with the slightly more restrictive definition of [3].) Such a shelling restricts to a shelling of $\partial K^* \setminus \text{star}(\hat{M})$, which therefore is a shellable $(d-1)$-ball.

With this the combinatorial argument of [4, Theorem 4.3] produces a (lexicographic) shelling of the face lattice $\{F \in L(K^*) : F \ngeq \hat{M}\} \,\dot{\cup}\, \{\hat{1}\}$ of $\partial K^* \setminus \text{star}(\hat{M})$. (Alternatively, one could use a geometric argument exploiting [7, §2(4)] to construct a simplicial polytope whose boundary complex is a suitable subdivision of ∂K^*, in order to produce a shelling of the poset $\{F \in \overline{L(K^*)} : F \ngeq \hat{M}\}$ geometrically. An argument of this type is used in [2, p. 174]. We omit details.) □

Note that Lemma 3.1 is not entirely trivial — in the category of triangulated or CW manifolds, the complement of a star in a sphere need *not* be a ball.

Theorem 3.2. *For every hyperplane arrangement $\mathscr{H}$, $L(\mathscr{H})$ is shellable.*

Proof. Let $\mathscr{H} = \{H_1, \ldots, H_n\}$ the hyperplane arrangement in $\mathbb{R}^d$. We interpret $\mathbb{R}^d$ as the affine subspace $\mathbb{R}^d \times \{1\}$ of $\mathbb{R}^{d+1}$. Now consider the central arrangement $\tilde{\mathscr{H}} = \{\tilde{H}_0, \tilde{H}_1, \ldots, \tilde{H}_n\}$, where for $1 \leqslant i \leqslant n$,

$$\tilde{H}_i = \operatorname{span}_{\mathbb{R}} H_i$$

and $\tilde{H}_0$ (corresponding to the "hyperplane at infinity" for $\mathscr{H}$) is given by

$$\tilde{H}_0 = \mathbb{R}^d \times \{0\} = \{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\}.$$

Again, we choose orthogonal unit vectors $\tilde{z}_0, \ldots, \tilde{z}_n$ to the hyperplanes in $\tilde{\mathscr{H}}$, and form the zonotope

$$\tilde{Z} = [-\tilde{z}_0, \tilde{z}_0] + \cdots + [-\tilde{z}_n, \tilde{z}_n]$$

and its polar dual $\tilde{Z}^*$.

We remark that the geometry of $\tilde{Z}$ is relevant for $\mathscr{H}$ because the closed faces of $\tilde{Z}$ contained in the "upper half space" $\mathbb{R}^d \times \mathbb{R}_0^+$ form a polyhedral complex combinatorially equivalent to a *dual block complex* for the dissection of $\mathbb{R}^d$ by $\mathscr{H}$ constructed in the spirit of [12, §64], whose face lattice is $L(\mathscr{H})^*$.

On the other hand, the cell decomposition of $\mathbb{R}^d$ by $\mathscr{H}$ is canonically isomorphic to the decomposition of the upper hemisphere ($x_{d+1} > 0$) of $S^d \subseteq \mathbb{R}^{d+1}$ by $\tilde{\mathscr{H}}$ and hence to the upper hemisphere of $\tilde{Z}^*$. If $\mathscr{H}$ is central, then $\tilde{Z}^*$ is a bipyramid over $Z^* = Z(\mathscr{H})^*$, and we get essentially nothing new.

Now consider the polytope

$$H^+ = \tilde{Z}^* \cap (\mathbb{R}^d \times \mathbb{R}_0^+) = \{z \in \tilde{Z}^* : z_{d+1} \geqslant 0\},$$

which is the "upper half" of $\tilde{Z}^*$. Its facets are the "bottom facet" contained in the supporting hyperplane $\tilde{H}_0$, plus the facets in the upper hemisphere of $\tilde{Z}^*$, which correspond to the regions (d-cells) of the dissection of $\mathbb{R}^d$ by $\mathscr{H}$. Hence the result follows from Lemma 3.1, applied to $K = H^+$, taking M to be the "bottom facet" of H^+. □

As a result of our discussion, we get the following strengthening of Lemma 2.1.

Corollary 3.3. *Let $\mathscr{H}$ be an (affine, regular) arrangement of hyperplanes in $\mathbb{R}^d$, then the cell decomposition induced on a large sphere $\lambda S^d = \{x \in \mathbb{R}^d : \|x\| = \lambda\}$ for $\lambda \gg 0$ is polytopal.*

Proof. A suitable polytope is the intersection of H^+ with a hyperplane that is parallel and sufficiently close to the "bottom facet" of H^+. □

There are several ways to study the topology of intervals in $L(\mathscr{H})$ for an affine arrangement $\mathscr{H}$. We will try to give a unified picture by comparing $L(\mathscr{H})$ with the face lattice of the convex polytope H^+.

Let $L(H^+)$ be this face lattice. Then $L(\mathscr{H})$ can (by the construction in Section 3) be seen as a sublattice of $L(H^+)$, where the faces of H^+ not in $L(\mathscr{H})$ are the bottom facet of H^+ lying in $\tilde{H}_0 = \mathbf{R}^d \times \{0\}$ and all its (non-empty) faces.

Hence, to understand the topology of intervals of $L(\mathscr{H})$, we can apply Lemma 3.1.

Theorem 3.4. *Let $F, G \in L(\mathscr{H}) \subset L(H^+)$ and $F < G$. If $F = \hat{0}$ and G is an unbounded cell in $\mathbf{R}^d$ or $G = \hat{1}$, then the open interval $\overline{(F; G)}$ is a shellable ball of dimension rank$(G) - 2 = \dim(G) - 1$. (In particular, $\overline{L(\mathscr{H})}$ is homeomorphic to a ball of dimension d.) In all other cases, $[F; G]$ is the face lattice of a (bounded convex) polytope and hence a shellable sphere of dimension $\dim(G) - \dim(F) - 2$.*

Proof. If $F = \hat{0}$ and G is an unbounded cell in $\mathbf{R}^d$ or $G = \hat{1}$, then the interval $[F; G]$ of $L(\mathscr{H})$ is the face lattice of the face $\tilde{G}$ of H^+ corresponding to G, with the proper faces of $\tilde{G}$ that lie in $\tilde{H}_0$ deleted. Its proper part is hence homeomorphic to a ball of dimension $\dim(G) - 1$ by Lemma 3.1.

In all other cases, $[F; G]$ is the face lattice of a (bounded convex) polytope (because the class of face lattices of convex polytopes is closed under taking intervals [8]) and hence homeomorphic to a sphere of dimension $\dim(G) - \dim(F) - 2$. □

4. Remarks

Theorems 3.2 and 3.4 imply Barnabei and Brini's main results, in particular both imply [1, Theorem 4.6]. Theorem 3.4 implies their computation of the Möbius function on $L(\mathscr{H})$ [1, Theorem 3.4].

In analogy to our study, one could discuss the face lattice $L(\mathscr{H})$ of dissections of projective space $\mathbf{R}P^n$ by hyperplane arrangements by reducing them to central arrangements. In this case, all *proper* intervals of $L(\mathscr{H})$ are face lattices of convex polytopes. However, $|\overline{L(\mathscr{H})}|$ is homeomorphic to $\mathbf{R}P^n$ and hence not shellable for $n \geq 2$.

Barnabei and Brini [1, Section 3] also study the dissection of convex sets $K \subseteq R^d$ by affine hyperplane arrangements. For this it is natural to consider

$$P_K(\mathscr{H}) = \{F \in P(\mathscr{H}) : F \cap K \neq \emptyset\}.$$

and $L_K(\mathscr{H}) = \{\hat{0}\} \cup P_K(\mathscr{H}) \cup \{\hat{1}\}$. Then $L_K(\mathscr{H})$ is a subposet of $L(\mathscr{H})$. If K is open, then $P_K(\mathscr{H})$ is a filter in $P(\mathscr{H})$. Even in this (well-behaved) case, however, $P_K(\mathscr{H})$ is not necessarily a graded poset, and hence a determination of its topology up to homeomorphism seems difficult.

The central case (Section 2) can be generalized to oriented matroids, which correspond to arrangements of "pseudo hyperplanes". For the face lattices of these structures, Edmonds and Mandel [9] and Lawrence [14] have shown that they are the face lattices of shellable spheres. However, the existence of polar duals heavily used in Section 3 does not generalize to oriented matroids [13], so that our geometric approach fails in this setting.

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